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Distance Regular Colorings of an Infinite Rectangular Grid

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Abstract—The parameters are listed of all distance regular colorings of the infinite rectangular grid.

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INTRODUCTION

A perfect coloring of vertices of a graph is characterized by the property that all vertices of the same color have the same color collection of their neighborhoods (the rigorous definitions will follow). The notion of *distance regular perfect coloring* (in other terminology, the *distance regular stratification of a graph*) is a rather useful tool for studying the invariant properties (say, *weighted distributions*) of various perfect structures. Going back to P. Delsarte [14] (see also [9]), it is repeatedly rediscovered and subjected to comprehensive study. Suffice it to say that, in a distance regular graph, the distance regular stratification of the set of its vertices, relative to any vertex, is a perfect coloring; and, moreover, the parameters of coloring are independent of the choice of a vertex.

Every perfect coloring with two colors is distance regular. Studying the various classes of graphs begins precisely with two colors. For an infinite rectangular grid, a full description of all perfect 2-colorings was obtained in [13], while the parameters of three-color colorings were described in [8].

Noticeable progress in the description of parameters of perfect 2-colorings for the Cayley graphs of an infinite cyclic group is reflected in [11]. For planar triangulations, partial investigation was done in [1]. The cases of a hypercube, a half-hypercube, and a Johnson graph were considered in [5–7, 10, 12]. Also, there is a series of results in this area concerning several infinite series of transitive cubic graphs [3].

The techniques of this work call to mind cristallization (or reconstruction of a full picture from some individual fragments) described in [2, 3, 15].

1. DEFINITIONS AND NOTATIONS

A vertex coloring of a graph with colors from 1 to k is called *perfect* if, for all not necessarily distinct $i, j = 1, 2, \dots, k$, there is an unambiguously defined integer α_{ij} equal to the number of vertices of color j in the neighborhood of each vertex of color i. The matrix (α_{ij}) is called the *matrix of parameters* of the coloring.

A perfect coloring is called *distance regular* if its matrix of parameters can be reduced to triangular form. In fact, this means that the colors in the coloring can be ordered so that each of them will see only two neighboring colors. The notion of distance regular coloring is directly connected with that of completely regular code in a graph. In our terminology, a *completely regular code* may be defined as the set of vertices of the first (or, last) color of a distance regular coloring. Thus, Theorem 1 enumerating the parameters of all distance regular colorings in $\mathbb{Z}^2$ implies the complete characterization of parameters of all these codes in $\mathbb{Z}^2$.

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2. INFINITE SERIES

Each coloring of an infinite n-dimensional grid can be thought of as a mapping

$$\varphi : \mathbb{Z}^n \to \{1, 2, \ldots, k\}.$$

In what follows, we will for convenience color in the rectangular grid its cells rather than vertices (the graph of the rectangular grid is self-dual).

Having a coloring $\varphi : \mathbb{Z}^n \to \{1, 2, \ldots, k\}$, it is easy to construct $\widehat{\varphi} : \mathbb{Z}^{(n+1)} \to \{1, 2, \ldots, k\}$ as follows:

$$\widehat{\varphi}(x_1, x_2, \ldots, x_{n+1}) = \varphi(x_1, x_2, \ldots, x_n).$$

Note that if φ is distance regular then so is $\widehat{\varphi}$; moreover, 2 is added to the diagonal elements in the matrix of parameters of the new coloring as compared to the old.

In general, there exist exactly three nonequivalent distance regular colorings of $\mathbb{Z}$ with $k \geq 2$ colors. They are periodic and their periods are $P_1 = 1, 2, \ldots, k, \ldots, 2$, $P_2 = 1, 1, 2, \ldots, k, \ldots, 2$, and $P_3 = 1, 1, 2, \ldots, k, k, \ldots, 2$. Applying to them the above procedures, we obtain the three series of colorings (consisting of the parallel one-color columns), and the three series of matrices of parameters, respectively:

$$A = \begin{pmatrix} 2 & 2 & . & . & . & 0 & 0 \\ 1 & 2 & 1 & . & . & . & 0 \\ 0 & 1 & 2 & 1 & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 1 & 2 & 1 \\ 0 & 0 & . & . & . & 2 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 & . & . & . & 0 & 0 \\ 1 & 2 & 1 & . & . & . & 0 \\ 0 & 1 & 2 & 1 & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 1 & 2 & 1 \\ 0 & 0 & . & . & . & 2 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 1 & . & . & . & 0 & 0 \\ 1 & 2 & 1 & . & . & . & 0 \\ 0 & 1 & 2 & 1 & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 1 & 2 & 1 \\ 0 & 0 & . & . & . & 1 & 3 \end{pmatrix}.$$

It is easy that from the one-dimensional distance regular colorings we can obtain new colorings as follows:

$$\widetilde{\varphi}(x_1, x_2, \ldots, x_n) = \varphi(x_1 + x_2 + \ldots + x_n).$$

Unlike the previous construction, this can be applied only to the one-dimensional colorings.

In the context of declared theme, we are only interested in the case $n = 2$. The obtained colorings consist of the parallel one-color diagonals, and the corresponding matrices D, E, and F are obtained from the matrices of one-dimensional colorings by doubling parameters:

$$D = \begin{pmatrix} 0 & 4 & . & . & . & 0 & 0 \\ 2 & 0 & 2 & . & . & . & 0 \\ 0 & 2 & 0 & 2 & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 2 & 0 & 2 \\ 0 & 0 & . & . & . & 4 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 2 & 2 & . & . & . & 0 & 0 \\ 2 & 0 & 2 & . & . & . & 0 \\ 0 & 2 & 0 & 2 & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 2 & 0 & 2 \\ 0 & 0 & . & . & . & 4 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 2 & 2 & . & . & . & 0 & 0 \\ 2 & 0 & 2 & . & . & . & 0 \\ 0 & 2 & 0 & 2 & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 2 & 0 & 2 \\ 0 & 0 & . & . & . & 2 & 2 \end{pmatrix}.$$

The above six infinite series give the distinct matrices with the exception of the coincidence of the matrices A and F for $k = 2$, to this matrix $\left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$ there correspond two distinct colorings.

Let us call the matrices of this section *reducible* since they are constructed by reduction to the one-dimensional case.

3. THE MAIN THEOREM

The main result of the article is

Theorem. *The matrices of parameters of distance regular colorings of the infinite rectangular grid are exhausted by the next list:*

(i) *six infinite series of reducible matrices;*

(ii) *four irreducible matrices of order* 2*:*

$$\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

(iii) *five irreducible matrices of order* 3*:*

$$\begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 3 \\ 0 & 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ 0 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 0 \\ 1 & 2 & 1 \\ 0 & 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

(iv) *two irreducible matrices of order* 4*:*

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

(v) *one irreducible matrix of order* 5*:*

$$\begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

Proof. Items (ii) and (iii) reduce to a simple choice of the triangular matrices (not belonging to infinite series) from the lists of the admissible ones in [8, 13]. The remaining items (cases of at least four colors) are proved by sorting of the possible variants of distribution of the first and fourth colors. In this connection, we considerably use the idea of unambiguous reconstruction of a perfect coloring from its fragments and available parameters. Sometimes, even if we do not know the parameters of a partial coloring, we can deduce its inconsistency if, for each its extension, some two identically colored cells have different color collections of the neighborhoods. Also, of use is the following obvious

Lemma. *Let some cells a and b at distance d in a distance regular coloring of $\mathbb{Z}^2$ have colors i and $i+d$. Then all shortest chains joining a and b are colored in a monotone increasing manner.*

Thus, consider a pair of vertices of the first and fourth colors situated at a possibly minimal distance from each other. If each pair lies at either vertical or horizontal line then we unambiguously have a coloring from one of the three infinite series A, B, or C.

Assume that there is a pair of vertices of the first and fourth colors joined by the chess horse move (Fig. 1). By Lemma 1, we can unambiguously determine two vertices of the second and two vertices of the third colors. For the vertex X, the possible variants are 1, 2, and 3 and for Y, the colors 2, 3, and 4. The possible pairs (X, Y) are as follows:

(1,4) gives the colorings from the series D, E, and F;

(3,4) gives the coloring of order 5 (Fig. 2);

(2,3) and (3,2) give the colorings of order 4 (Fig. 3).
The remaining variants are inconsistent. Let us prove this.

Case (1,2) (Fig. 4). This case is most complicated. In Fig. 4p, we denote by letters the cells whose colors are unambiguously determined by Lemma 1, and also by the information of the color structure of neighborhoods of vertices. The reconstruction goes in alphabetical order and leads to the situation in Fig. 4b. It is easy that, for the vertex z, the only possible variant of coloring is the color 0 (adding this color leads really to the coloring in 5 colors), but by assumption the color 1 is minimal; a contradiction.

	Y	
2	3	4
1	2	3
	X	

Fig. 1

2	3	4	3	2	3	4	3
3	4	**5**	4	3	4	**5**	4
2	3	4	3	2	3	4	3
1	2	3	2	**1**	2	3	2
2	3	4	3	2	3	4	3
3	4	**5**	4	3	4	**5**	4
2	3	4	3	2	3	4	3
1	2	3	2	**1**	2	3	2

Fig. 2

3	**4**	3	2	3	2	**1**	2
2	3	2	**1**	2	3	2	3
1	2	3	2	3	**4**	3	2
2	3	**4**	3	2	3	2	**1**
3	2	3	2	**1**	2	3	2
2	**1**	2	2	2	3	**4**	3
3	2	3	**4**	3	2	3	2
4	3	2	3	2	**1**	2	3

4	3	2	3	**4**	3	2	3
4	3	2	3	**4**	3	2	3
3	2	**1**	2	2	2	**1**	2
3	2	**1**	2	2	2	**1**	2
4	3	2	3	**4**	3	2	3
4	3	2	3	**4**	3	2	3
3	2	**1**	2	2	2	**1**	2
3	2	**1**	2	2	2	**1**	2

Fig. 3

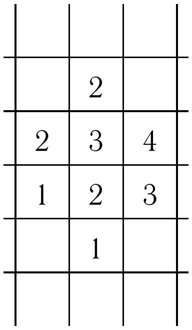

Fig. 4

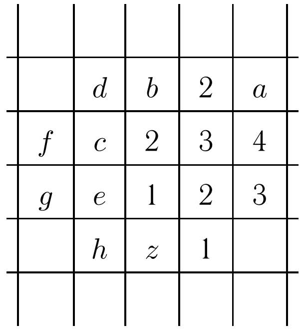

Fig. 4a

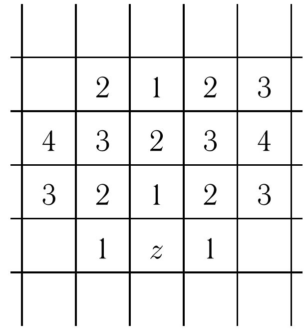

Fig. 4b

Case (3,3) (Fig. 5). It is seen in Fig. 5 that the vertex z can be colored only with 2 or 3, but each of these immediately leads to a contradiction. The selected vertices of color 2, in the first case, and of 3, in the second, necessarily have the distinct color structure of the neighborhoods.

Case (2,4) (Fig. 6) is similar to (3,3), we leave it to the reader. In the case (2,2) (Fig. 7), every color of the cell z is inconsistent since the inner degrees of colors 2 and 3 cannot be unambiguously determined. In the last case (1,3) (Fig. 8), we only say that it is as much symmetric to the case (2,2) as the case (3,3) to (2,4). □

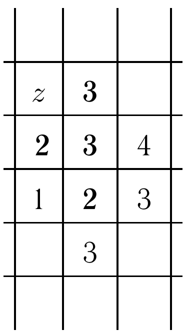

Fig. 5

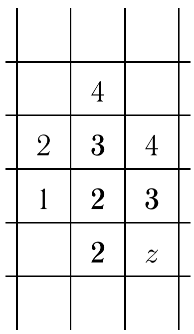

Fig. 6

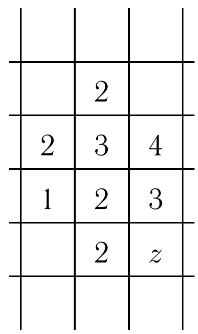

Fig. 7

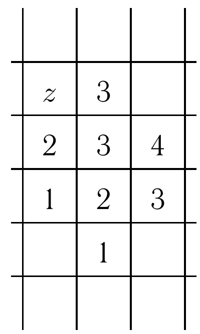

Fig. 8

4. CONCLUSION

Note that in not all cases the matrix unambiguously determines the coloring. We can say that the matrix $\begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$ accepts two nonequivalent colorings, while the matrices $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$, uncountably many at all. A good luck for the authors is the fact that the principal difficulties in describing the distance regular colorings of the rectangular grid are condensed at the two color and three color cases which were completely studied earlier in [13] and [8] respectively. In the case of a greater number of colors, there corresponds a unique (up to equivalence) coloring to each admissible matrix of parameters.

We emphasize that it is not often encounter the research which succeeded in giving comprehensive description of the structure of completely regular codes. A hope might arise that, in the grids of greater dimensions, some progress is possible too. First of all, this is connected with the hypothesis on the finiteness of a number of the irreducible admissible matrices for each fixed dimension.

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